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The Asymptotic Stability of a Nonstationary System with Delay

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Abstract—In this paper we study the exponential stability of a linear system of differential equations with a constant delay such that the right-hand side of one of its subsystems contains an exponential multiplier. We establish sufficient conditions for the existence of an asymptotically periodic solution to the inhomogeneous system.

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1. STABILITY OF NONSTATIONARY HOMOGENEOUS SYSTEMS

Consider the following linear homogeneous system with a constant delay:

$$\begin{gathered} dx(t)/dt = A_1(t)x(t) + A_2(t)x(t-\sigma) + B_1(t)y(t) + B_2(t)y(t-\sigma), \\ dy(t)/dt = \theta_0 e^t[A_3x(t) + A_4x(t-\sigma) + B_3y(t) + B_4y(t-\sigma)], \\ t \geq 0,\ \ \theta_0 > 0,\ \ \sigma = \mathrm{const},\ \ \sigma > 0. \end{gathered} \tag{1.1}$$

Here $A_i(t)$ and $B_i(t)$ $(i = 1, 2)$ are periodic (with a period $0 < T \leq \sigma$) continuously differentiable $m \times m$-matrices, A_k and B_k $(k = 3, 4)$ are constant $m \times m$-matrices, while $x(t)$ and $y(t)$ are m-dimensional vector functions with respect to (the argument) t.

We define the norm of a vector $w = \{w_j\}^{(\top)}$ (here w_j are components of the vector w, and $(^\top)$ is the transposition sign) by the equality $\|w\| = \sum\limits_{j=1}^{m} |w_j|$; we do the norm of a matrix $D = \{d_{ij}\}$ $(i, j = 1, \ldots, m)$ in accordance with the norm of a vector ([1], P. 12), namely, $\|D\| = \max\limits_j \sum\limits_i |d_{ij}|$.

Earlier we studied properties of a system with constant matrices A_i^0 and B_i^0 $(i = 1, 2)$ in papers [2–4]. Note that one can obtain system (1.1) by replacing the time variable $t = \ln(\frac{\theta}{\theta_0})$ in the following system with a linear delay:

$$\begin{gathered} d\widehat{x}(\theta)/d\theta = \frac{1}{\theta}[\widehat{A}_1(\theta)\widehat{x}(\theta) + \widehat{A}_2(\theta)\widehat{x}(\mu\theta) + \widehat{B}_1(\theta)\widehat{y}(t) + \widehat{B}_2(\theta)\widehat{y}(\mu\theta)], \\ d\widehat{y}(\theta)/d\theta = A_3\widehat{x}(\theta) + A_4\widehat{x}(\mu\theta) + B_3\widehat{y}(\theta) + B_4y(\mu\theta), \\ \theta \geq \theta_0 > 0,\ \ \sigma = -ln(\mu),\ \ 0 < \mu < 1. \end{gathered}$$

On an initial set $[\mu\theta_0, \theta_0]$ we define an initial vector function $\widehat{\phi}(\theta)$. We can solve this system by successive integration on segments $h_n(\theta) = \left(\frac{\theta_0}{\mu^n}, \frac{\theta_0}{\mu^{n+1}}\right]$. This method does not allow one to establish asymptotic properties of the system under consideration.

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As was proved earlier, solution $\{x^0(t), y^0(t)\}^\top$ to the system

$$dx^0(t)/dt = A_1^0 x^0(t) + A_2^0 x^0(t-\sigma) + B_1^0 y^0(t) + B_2^0 y^0(t-\sigma),$$
$$dy^0(t)/dt = \theta_0 e^t [A_3 x^0(t) + A_4 x^0(t-\tau) + B_3 y^0(t) + B_4 y^0(t-\tau)], \quad t \geq 0, \tag{1.2}$$

is exponentially stable, and the system itself is stable in the first approximation, if all the following conditions are fulfilled:
1) roots λ of the characteristic equation

$$|A_1^0 + A_2^0 e^{-\lambda\sigma} - \lambda E| = 0$$

have negative real parts, i.e.,

$$\operatorname{Re}\lambda < -\beta_1, \ \ \beta_1 = \text{const}, \ \ \beta_1 > 0; \tag{1.3}$$

2) eigenvalues ν of the matrix B_3 have negative real parts, i.e.,

$$\operatorname{Re}\nu < -\beta_2, \ \ \beta_2 = \text{const}, \ \ \beta_2 > 0; \tag{1.4}$$

3) roots p of the characteristic equation

$$\left|B_3 + B_4 e^{-p\sigma} - \left(A_3 + A_4 e^{-p\sigma}\right)\left(A_1^0 + A_2^0 e^{-p\sigma} - pE\right)^{-1}\left(B_1^0 + B_2^0 e^{-p\sigma}\right)\right| = 0$$

satisfy the inequality

$$\operatorname{Re}\nu < -\beta_3, \ \ \beta_3 = \text{const}, \ \ \beta_3 > 0. \tag{1.5}$$

In the paper [2] with the help of the Laplace transform [5] we prove that under conditions (1.3)–(1.5) solution $x^0(t, \phi_1(\eta))$, $y^0(t, \phi_2(\eta))$ to system (1.2) (defined by the initial vector function $\phi^{(\top)}(\eta) = \{\phi_1(\eta), \phi_2(\eta)\} : x(\eta) = \phi_1(\eta)$, $y(\eta) = \phi_2(\eta)$, $\eta \in [-\sigma, 0]$) satisfies the bound

$$\|x^0(t)\| + \|y^0(t)\| \leq M_0 e^{-\beta_0 t}[\sup_\eta \|\phi_1(\eta)\| + \sup_\eta \|\phi_2(\eta)\|], \ \ t \geq 0, \tag{1.6}$$
$$M_0 = \text{const}, \ \ M_0 > 1, \ \ \beta_0 = \min\{\beta_1, \beta_3\} - \overline{\varepsilon}$$

($\beta_0 > 0$ with sufficiently small positive $\overline{\varepsilon}$).

Let us define a solution to the initial system (1.1) by the same vector function $\phi(\eta)$ (we set $\sup_\eta \|\phi_1(\eta)\| + \sup_\eta \|\phi_2(\eta)\| < \delta_0$, while δ_0 is a sufficiently small positive number). In order to better understand properties of the initial system, we proceed from system (1.1) (a finite system of differential equations with a delay on an infinite time interval) to a countable system of ordinary differential equations given on a finite time interval $[0, \sigma]$. To this end, we set $x_{n+1}(t) = x(n\sigma + t)$ and $y_{n+1}(t) = y(n\sigma + t)$ $(n = 1, 2 \ldots)$, $t \in [0, \sigma]$). We get the following totality of two subsystems:

$$dx_{n+1}(t)/dt = A_1(t)x_{n+1}(t) + A_2(t)x_n(t) + B_1(t)y_{n+1}(t) + B_2(t)y_n(t), \tag{1.7}$$

$$\varepsilon_n dy_{n+1}(t)/dt = e^t[A_3 x_{n+1}(t) + A_4 x_n(t) + B_3 y_{n+1}(t) + B_4 y_n(t)], \tag{1.8}$$

$$t_0 \leq t \leq t_0 + \sigma, \ \ \varepsilon_n = e^{-n\sigma}/\theta_0 = \mu^n/\theta_0, \ \ x_0(t) = \phi_1(t-\sigma), \ \ y_0 = \phi_2(t-\sigma),$$

satisfying boundary conditions ([5], P. 103)

$$x_{n+1}(0) = x_n(\sigma), \ \ y_{n+1}(0) = y_n(\sigma).$$

Thus, we have reduced the solution of system (1.1) to the successive integration of a "differential-difference" system in the space of continuous vector functions given on the segment $[0, \sigma]$. Let us mention some properties of subsystem (1.8). Since $\varepsilon_n \to 0$ as $n \to \infty$, subsystem (1.8) is singularly perturbed [2, 3], consequently, the solution set of system (1.7), (1.8) contains slow variables ($x_n(t)$) and fast ones ($y_n(t)$). Moreover, with sufficiently large n, just the presence of a small parameter ε_n allows us to consider the asymptotic behavior of subsystem (1.7) with periodic matrices $A_i(t)$ and $B_i(t)$ $(i = 1, 2)$.

It is well-known ([6], P. 484) that periodic continuously differentiable matrices $A_k(t)$ and $B_k(t)$ are representable as absolutely convergent Fourier series, namely,

$$A_k(t) = \frac{\alpha_0^k}{2} + \sum_{j=1}^{\infty}[\alpha_j^k \cos(jrt) + \beta_j^k \sin(jrt)],$$

$$B_k(t) = \frac{\widehat{\alpha}_0^k}{2} + \sum_{j=1}^{\infty}[\widehat{\alpha}_j^k \cos(jrt) + \widehat{\beta}_j^k \sin(jrt)], \ \ k = 1, 2; \ \ r = 2\pi/\sigma.$$

In view of the absolute convergence of these series, there exists a positive integer number $\widehat{k}$ satisfying approximate equalities

$$\begin{aligned} A_k(t) &\approx \frac{\alpha_0^k}{2} + \sum_{j=1}^{\widehat{k}}[\alpha_j^k \cos(jrt) + \beta_j^k \sin(jrt)] = \widehat{A}_k(t), \\ B_k(t) &\approx \frac{\widehat{\alpha}_0^k}{2} + \sum_{j=1}^{\widehat{k}}[\widehat{\alpha}_j^k \cos(jrt) + \widehat{\beta}_j^k \sin(jrt)] = \widehat{B}_k(t), \ \ k = 1, 2. \end{aligned} \tag{1.9}$$

By replacing the time variable $t = \varepsilon_n \tau_n$ in system (1.7), (1.8), taking into account approximations (1.9), we obtain

$$\begin{aligned} dx_{n+1}(\tau_n)/d\tau_n = \varepsilon_n[\widehat{A}_1(\varepsilon_n\tau_n)x_{n+1}(\tau_n) + \widehat{A}_2(\varepsilon_n\tau_n)x_n(\tau_n) \\ + \widehat{B}_1(\varepsilon_n\tau_n)y_{n+1}(\tau_n) + \widehat{B}_2(\varepsilon_n\tau_n)y_n(\tau_n)], \end{aligned} \tag{1.10}$$

$$dy_{n+1}(\tau_n)/d\tau_n = e^{\varepsilon_n\tau_n}\left[A_3x_{n+1}(\tau_n) + A_4x_n(\tau_n) + B_3y_{n+1}(\tau_n) + B_4y_n(\tau_n)\right], \tag{1.11}$$

$$0 \le \tau_n \le \frac{\sigma}{\varepsilon_n}, \ \ \varepsilon_n = \frac{e^{-n\sigma}}{\theta_0}, \ \ \lim_{n\to\infty}\varepsilon_n = 0, \ \ \sum_{j=0}^{\infty}\varepsilon_j < \infty.$$

Consider subsystem (1.10) with sufficiently large n. As is known ([7], P. 177), one can treat the totality of nonstationary terms in the right-hand side of this subsystem as linear perturbations "oscillating" with sufficiently small frequency, whose norms satisfy, respectively, bounds ([8], P. 259)

$$\int_s^{\tau}\delta_j(s)ds \le \widehat{\varepsilon}_n(\tau - s), \lim_{n\to\infty}\widehat{\varepsilon}_n = 0, \ \ j = 1, 2, \tag{1.12}$$

where

$$\delta_1(\tau_n) = \|x_{n+1}(\tau_n)\|, \ \ \delta_1(\tau_n) = \|y_{n+1}(\tau_n)\|, \ \ \delta_2(\tau_{n-1}) = \|x_n(\tau_n)\|, \ \ \delta_2(\tau_{n-1}) = \|y_n(\tau_n)\|.$$

Consequently, the first approximation subsystem for (1.10) with sufficiently large n takes the form

$$dx^0_{n+1}(\tau_n)/d\tau_n = \varepsilon_n\left\{\frac{\alpha_0^1}{2}x^0_{n+1}(\tau_n) + \frac{\alpha_0^2}{2}x^0_n(\tau_n) + \frac{\alpha_0^3}{2}y^0_{n+1}(\tau_n) + \frac{\alpha_0^4}{2}y^0_n(\tau_n)\right\},$$

and the first approximation system corresponding to (1.10), (1.11) is the following differential-difference system

$$\begin{aligned} dx^0_{n+1}(\tau_n)/d\tau_n &= \varepsilon_n\left\{\frac{\alpha_0^1}{2}x^0_{n+1}(\tau_n) + \frac{\alpha_0^2}{2}x^0_n(\tau_n) + \frac{\alpha_0^3}{2}y^0_{n+1}(\tau_n) + \frac{\alpha_0^4}{2}y^0_n(\tau_n)\right\}, \\ dy^0_{n+1}(\tau_n)/dt &= e^{\varepsilon_n\tau_n}\left\{A_3x^0_{n+1}(\tau_n) + A_4x^0_n(\tau_n) + B_3y^0_{n+1}(\tau_n) + B_4y^0_n(\tau_n)\right\}. \end{aligned} \tag{1.13}$$

Now let system (1.13) satisfy exponential stability conditions, i.e., inequalities (1.3), (1.4), and (1.5).

Theorem 1. *If constant ("averaged") matrices α_0^k, $k = 1, \dots, 4$, satisfy conditions* (1.3), (1.4), *and* (1.5)*, then system* (1.7), (1.8) *is exponentially stable.*

Proof. Let us write the solution to system (1.10), (1.11) in the operator form

$$\begin{pmatrix} x^0_{n+1}(\tau_n) \\ y^0_{n+1}(\tau_n) \end{pmatrix} = T_{\tau_n,n}\begin{pmatrix} x^0_n(\tau_n), y^0_{n+1}(\tau_n), y^0_n(\tau_n) \\ y^0_n(\tau_n), x^0_{n+1}(\tau_n), x^0_n(\tau_n) \end{pmatrix},$$

where the linear shift operator obeys the formula

$$T_{\tau_n,n} = \begin{Bmatrix} T^1_{\tau_n,n} \\ T^2_{\tau_n,n} \end{Bmatrix} = \begin{cases} U_n(\tau_n)u_n(\frac{\sigma}{\varepsilon_n}) + \int\limits_0^{\tau_n} U_n(\tau_n - s)0.5\varepsilon_n\alpha_0^2 u_n(s)ds \\ \qquad + \int\limits_0^{\tau_n} U_n(\tau_n - s)0.5\varepsilon_n(\alpha_0^3 v_{n+1}(s) + \alpha_0^4 v_n(s))ds, \\ V_n(\tau_n)v_n(\frac{\sigma}{\varepsilon_n}) + \int\limits_0^{\tau_n} V_n(\tau_n, s)e^{\varepsilon_n s}(B_4 v_n(s) \\ \qquad + A_3 u_{n+1}(s) + A_4 u_n(s))ds. \end{cases} \tag{1.14}$$

Here $U_n(\tau - s)$ is the fundamental solution matrix of the homogeneous system

$$d\overline{u}/d\tau = 0.5\varepsilon_n[\alpha_0^1\overline{u}(\tau) + \alpha_0^2\overline{u}(\tau - \sigma)],\ \ 0 \le \tau \le \sigma/\varepsilon_n;$$

owing to inequality (1.5)), it satisfies the bound

$$\|U_n(\tau - s)\| \le M_1 e^{-\varepsilon_n\beta_1(\tau - s)},\ \ M_1 > 0. \tag{1.15}$$

Analogously, the matrix

$$V_n(\tau, s) = \exp\left\{B_3\frac{e^{\varepsilon_n\tau} - e^{\varepsilon_n s}}{\varepsilon_n}\right\}$$

is the fundamental solution matrix of the homogeneous system

$$d\overline{v}/d\tau = e^{\varepsilon_n\tau}B_3\overline{v}(\tau),$$

and it also satisfies the bound

$$\|V_n(\tau, s)\| \le M_2\exp\left\{\frac{-\beta_2}{\varepsilon_n}(e^{\varepsilon_n\tau} - e^{\varepsilon_n s})\right\},\ \ M_2 \ge 1. \tag{1.16}$$

Note that in view of bounds (1.15) and (1.16) the given operator $T_{\tau_n,n}$ is uniformly bounded [3, 4], i.e., $\|T_{\tau_n,n}\| \le \overline{M}$, $\overline{M} > 1$. Inequality (1.6) for the product of operators $T_{n_j,j}$ implies the bound [4]

$$\left\|T_{\tau_n,n}\prod_{j=0}^{n-1}T_{s_j,j}w_0(s)\right\| \le \widehat{M}_0 q^n(\sup_\eta\|x_0(\eta)\| + \sup_\eta\|y_0(\eta)\|),$$

$$w_0^{(\top)}(\eta) = \{x_0(\eta), y_0(\eta)\},\ \ \eta\in[-\sigma, 0],\ \ q = e^{-\beta_0\sigma},\ \ 0 < q < 1,\ \ \widehat{M}_0 > 1.$$

Assume that ε_n is sufficiently small for $n \ge N$ (N is a positive integer number).

Let us seek for a solution to the "perturbed" system (1.10), (1.11) stepwise. Denote "perturbations", respectively, by

$$f^1_{n+1}(\tau_n, x_{n+1}(\tau_n), x_n(\tau_n)) = \big(\widehat{A}_1(\tau_n) - \alpha_0^1/2\big)x_{n+1}(\tau_n) + \big(\widehat{A}_2(\tau_n) - \alpha_0^2/2\big)x_n(\tau_n),$$
$$f^2_{n+1}(\tau_n, y_{n+1}(\tau_n), y_n(\tau_n)) = \big(\widehat{B}_1(\tau_n) - \widehat{\alpha}_0^1/2\big)y_{n+1}(\tau_n) + \big(\widehat{B}_2(\tau_n) - \widehat{\alpha}_0^2/2\big)y_n(\tau_n).$$

We write the system solution in the integral (Cauchy) form, treating perturbing (oscillating) terms as inhomogeneities ([1], P. 162), and represent the solution in the operator form

$$\begin{pmatrix} x_{n+1}(\tau_n) \\ y_{n+1}(\tau_n) \end{pmatrix} = T_{\tau_n,n}\begin{pmatrix} x_n^0(\tau_n), y_{n+1}^0(\tau_n), y_n^0(\tau_n) \\ y_n^0(\tau_n), x_{n+1}^0(\tau_n), x_n^0(\tau_n) \end{pmatrix}$$
$$+ I_n\left(f^1_{n+1}(\tau_n, x_{n+1}(\tau_n), x_n(\tau_n)), f^2_{n+1}(\tau_n, y_{n+1}(\tau_n), y_n(\tau_n))\right).$$

Here I_n is the linear integral operator defined as

$$I_n^1(f^1_{n+1}(\tau_n, x_{n+1}(\tau_n), x_n(\tau_n)), f^2_{n+1}(\tau_n, y_{n+1}(\tau_n), y_n(\tau_n)))$$
$$= \int_0^{\tau_n} U_n(\tau_n - s_n)\varepsilon_n f^1_{n+1}(s_n, x_{n+1}(s_n), x_n(s_n))ds_n$$

$$+\int_0^{\tau_n} U_n(\tau_n - s_n)\varepsilon_n f^2_{n+1}(s_n, y_{n+1}(s_n), y_n(s_n))ds_n.$$

Taking into account bounds (1.12), let us prove that terms containing values $f^1_{n+1}(\tau_n, x_{n+1}(\tau_n), x_n(\tau_n))$ and $f^2_{n+1}(\tau_n, y_{n+1}(\tau_n), y_n(\tau_n))$, have a higher order of smallness than unperturbed terms. Assume that $\tau_n > 1$, $0 \le s_n \le k + \vartheta$, k is a positive integer number, $\vartheta = \mathrm{const}$, and $0 < \vartheta < 1$. Then we get the following chain of inequalities:

$$\begin{aligned}
&\|I^1_n(f^1_{n+1}(\tau_n, x_{n+1}(\tau_n), x_n(\tau_n)))\| \\
&\quad\le \int_0^{\tau_n} M_1 e^{-\varepsilon_n\beta_1(\tau_n - s_n)}\varepsilon_n\delta_1(s_n)\|x_{n+1}(s_n)\|ds_n + \int_0^{\tau_n} M_1 e^{-\varepsilon_n\beta_1(\tau_n - s_n)}\varepsilon_n\delta_2(s_n)\|x_n(s_n)\|ds_n \\
&\qquad\le M_1 e^{-\varepsilon_n\beta_i\tau_n}\sup_{0\le s_n\le\tau_n}\|x_{n+1}(s_n)\|\sum_{j=1}^{k+1} e^{\varepsilon_n\beta_1 j}\int_j^{j+1}\varepsilon_n\delta_1(r)dr \\
&\qquad\quad + M_1 e^{-\varepsilon_n\beta_1\tau_n}\sup_{0\le s_n\le\tau_n}\|x_n(s_n)\|\sum_{j=1}^{k+1} e^{\varepsilon_n\beta_1 j}\int_j^{j+1}\varepsilon_n\delta_2(r)dr \\
&\qquad\qquad \le \varepsilon_n\widehat{\varepsilon}_n M_1\frac{e^{\varepsilon_n\beta_1}}{e^{\varepsilon_n\beta_1}-1}\sup_{0\le s_n\le\tau_n}(\|x_{n+1}(s_n)\| + \|x_n(s_n)\|).
\end{aligned}$$

Note that $\lim\limits_{n\to\infty}\frac{\varepsilon_n}{e^{\varepsilon_n\beta_1}-1} = \frac{1}{\beta_1}$.

If $\tau \le 1$, then

$$\begin{aligned}
&M_1\int_0^s e^{-\varepsilon_n\beta_1(\tau_n - s_n)}\varepsilon_n\delta_1(s_n)\|x_{n+1}(s_n)\|ds_n + M_1\int_0^s e^{-\varepsilon_n\beta_1(\tau_n - s_n)}\varepsilon_n\delta_2(s_n)\|x_n(s_n)\|ds_n \\
&\qquad\le M_1\varepsilon_n\widehat{\varepsilon}_n\big(\sup_{0\le s_n\le\tau_n}\|x_{n+1}(s_n)\| + \sup_{0\le s_n\le\tau_n}\|x_{n+1}(s_n)\|\big).
\end{aligned}$$

In both cases the sum of these integrals does not exceed the value

$$\mathbf{O}(\widehat{\varepsilon}_n)\big(\sup_{0\le s_n\le\tau_n}\|x_{n+1}(s_n)\| + \sup_{0\le s_n\le\tau_n}\|x_n(s_n)\|\big). \tag{1.17}$$

Now, estimating the value $\|I^1_n(f^2_{n+1}(\tau_n, y_{n+1}(\tau_n), y_n(\tau_n)))\|$, we obtain an analogous bound, namely,

$$\|I^1_n(f^2_{n+1}(\tau_n, y_{n+1}(\tau_n), y_n(\tau_n)))\| = \mathbf{O}(\widehat{\varepsilon}_n)\big(\sup_{0\le s_n\le\tau_n}\|y_{n+1}(s_n)\| + \sup_{0\le s_n\le\tau_n}\|y_n(s_n)\|\big). \tag{1.18}$$

Since with $n < N$ system (1.10) (1.11) has bounded right-hand sides satisfying the Lipschitz conditions, we can estimate its solution with the help of the Bellman–Gronwall lemma ([8], P. 517). This solution $\{x_{n-1}, y_{n-1}\}^\top$ is, at least, bounded, i.e., there exists $L > 0$ such that

$$\max_{\tau_{n-2}}(\|x_{n-1}(\tau_{n-2})\| + \|y_{n-1}(\tau_{n-2})\|) \le L\sup_\eta(\|\phi_1(\eta)\| + \|\phi_2(\eta)\|). \tag{1.19}$$

For the solution to the perturbed system (1.10), (1.11) defined by the initial vector function $\{x_{n-1}(\tau_{n-2}), y_{n-1}(\tau_{n-2})\}^\top$ with $n \ge N$, in view of bounds (1.17) and (1.18) (i.e., the smallness of perturbations), taking into accounts results obtained by us in [4], we get the bound

$$\max_{\tau_{n+j}}(\|x_{n+j+1}(\tau_{n+j})\| + \|y_{n+j+1}(\tau_{n+j})\|) \le L_1(q_1)^j\max_{\tau_{n-2}}(\|x(\tau_{n-2})\| + \|y(\tau_{n-2})\|), \tag{1.20}$$

$$q < q_1 < 1, \quad L_1 > 1.$$

Formulas (1.19) and (1.20) imply an analogous exponential bound for any $x_n(\tau_{n-1})$, $y_n(\tau_{n-1})$, $n = 1, 2, \ldots$. Furthermore, since

$$\|A_k(t) - \widehat{A}_k(t)\| < \varepsilon, \quad \|B_k(t) - \widehat{B}_k(t)\| < \varepsilon, \quad k = 1, 2$$

(ε is a sufficiently small positive number), the exponential stability of the first approximation system (1.13) implies the same property of system (1.10), (1.11) (in view of results obtained in [4]). □

2. THE EXISTENCE OF AN ASYMPTOTICALLY PERIODIC SOLUTION

Consider the inhomogeneous system

$$\begin{gathered} d\widehat{x}(t)/dt = A_1(t)\widehat{x}(t) + A_2(t)\widehat{x}(t-\sigma) + B_1(t)\widehat{y}(t) + B_2(t)\widehat{y}(t-\sigma) + f_1(t), \\ d\widehat{y}(t)/dt = \theta_0 e^t[A_3\widehat{x}(t) + A_4\widehat{x}(t-\sigma) + B_3\widehat{y}(t) + B_4\widehat{y}(t-\sigma) + f_2(t)], \\ t \geq 0,\ \ \theta_0 > 0,\ \ \sigma > 0. \end{gathered} \tag{2.1}$$

Here $f(t)$ $(j = 1, 2)$ are periodic continuously differentiable functions with the period $T = \sigma$. Assume that the corresponding homogeneous system satisfies conditions of Theorem 1. Then the solution to the perturbed homogeneous system (that corresponds to the first approximation system (1.13)) satisfies the bound

$$\|x(\tau_n)\| + \|y(\tau_n)\| \leq \widehat{L}\prod_{j=0}^{n}(q + \|\mathbf{O}(\varepsilon_j)\| + \|\mathbf{O}(\varepsilon)\|)(\sup_\eta \|\phi_1(\eta)\| + \sup_\eta \|\phi_2(\eta)\|), \tag{2.2}$$

$$\lim_{j\to\infty}(q + \|\mathbf{O}(\varepsilon_j)\| + \|\mathbf{O}(\varepsilon)\|) = q_1,\ \ q < q_1 < 1,\ \ \sum_{j=1}^{\infty}\|\mathbf{O}(\varepsilon_j)\| < \infty.$$

Taking into account the asymptotic behavior of the solution to the inhomogeneous perturbed system written in the integral form stepwise, in view of bound (2.2) we obtain

$$\begin{aligned} \|\widehat{x}(\tau_n)\| + \|\widehat{y}(\tau_n)\| &\leq \widehat{L}\prod_{j=0}^{n}(q + \|\mathbf{O}(\varepsilon_j)\| + \|\mathbf{O}(\varepsilon)\|)\big(\sup_\eta \|\phi_1(\eta)\| + \sup_\eta \|\phi_2(\eta)\|\big) \\ &+ \widehat{L}\prod_{j=0}^{n-1}(q + \|\mathbf{O}(\varepsilon_j)\| + \|\mathbf{O}(\varepsilon)\|)\big(\max_{0\leq t\leq\sigma}\|f_1(t)\| + \max_{0\leq t\leq\sigma}\|f_2(t)\|\big) \\ &+ \widehat{L}\prod_{j=0}^{n-2}(q + \|\mathbf{O}(\varepsilon_j)\| + \|\mathbf{O}(\varepsilon)\|)\big(\max_{0\leq t\leq\sigma}\|f_1(t)\| + \max_{0\leq t\leq\sigma}\|f_2(t)\|\big) + \cdots \\ &+ \widehat{L}\big(\max_{0\leq t\leq\sigma}\|f_1(t)\| + \max_{0\leq t\leq\sigma}\|f_2(t)\|\big) \leq \widehat{L}_1\prod_{j=0}^{n}(q_1 + \|\mathbf{O}(\varepsilon_j)\|)\big(\sup_\eta \|\phi_1(\eta)\| + \sup_\eta \|\phi_2(\eta)\|\big) \\ &+ \frac{\widehat{L}_1}{1-q_1}\big(\max_{0\leq t\leq\sigma}\|f_1(t)\| + \max_{0\leq t\leq\sigma}\|f_2(t)\|\big),\ \ \widehat{L} < \widehat{L}_1 < \infty. \end{aligned}$$

This estimate means the boundedness of solutions $\{\|\widehat{x}(t, \phi_1(\theta))\|, \|\widehat{y}(t, \phi_2(\theta))\|\}$ to system (2.1). Then (as follows from Theorem 1, namely, from a correlation analogous to (1.10)) $\lim\limits_{t\to\infty} d\widehat{x}(t)/dt = 0$. This allows us to study asymptotic properties of some partial solutions under additional conditions imposed on the right-hand sides of the inhomogeneous system (2.1).

The next theorem is a continuation of the study performed in [9].

Theorem 2. *Let assumptions of Theorem* 1 *be fulfilled together with the following conditions*:
1) *there exists a matrix* $(B_3 + B_4)^{-1}$,
2) *there exists a constant* m*-dimensional vector* $\widehat{C}$ *such that*

$$(A_1(t) + A_2(t))\widehat{C} + f_1(t) = (B_1(t) + B_2(t))\big[(B_3 + B_4)^{-1}[(A_3 + A_4)\widehat{C} + f_2(t)]\big].$$

Then system (2.1) *has an asymptotically periodic* (*with a period* σ) *solution* $\{\widehat{x}_\sigma, \widehat{y}_\sigma\}^\top$, *where*

$$\widehat{x}_\sigma = \widehat{C},\ \ \widehat{y}(t) = -(B_3 + B_4)^{-1}[(A_3 + A_4)\widehat{C} + f_2(t)].$$

Proof. System (2.1) corresponds to the inhomogeneous differential-difference system

$$\begin{aligned} d\widehat{x}_{n+1}(t)/dt &= A_1(t)\widehat{x}_{n+1}(t) + A_2(t)\widehat{x}_n(t) + B_1(t)\widehat{y}_{n+1}(t) + B_2(t)\widehat{y}_n(t) + f_1(t), \\ \varepsilon_n d\widehat{y}_{n+1}(t)/dt &= e^t[A_3\widehat{x}_{n+1}(t) + A_4\widehat{x}_n(t) + B_3\widehat{y}_{n+1}(t) + B_4\widehat{y}_n(t) + f_2(t)], \end{aligned} \tag{2.3}$$

$$0 \le t \le \sigma, \;\; \widehat{y}_{n+1}(0) = \widehat{y}_n(\sigma), \;\; \widehat{x}(t) \approx \widehat{C}, \;\; n \ge N.$$

Consider the second subsystem. Let N be a sufficiently large positive integer number. Because of the presence of the small parameter ε_n in the left-hand side of the subsystem, we seek for a partial solution in the form

$$\widehat{y}_\sigma(t) = -(B_3 + B_4)^{-1}[(A_3 + A_4)\widehat{C} + f_2(t)]. \tag{2.4}$$

Let $\widehat{x}_n(t) = \widehat{C} + x_n^0(t)$ and $\widehat{y}_n(t) = y(t)_\sigma + y_n^0(t)$. Then in view of (2.4) from the second subsystem in (2.3) we get

$$dx_{n+1}(t)^0/dt = A_1(t)x_{n+1}^0(t) + A_2(t)x_n^0(t) + B_1(t)y_{n+1}^0(t) + B_2(t)y_n^0(t),$$

$$\begin{aligned} &\varepsilon_n dy_\sigma(t)/dt + \varepsilon_n dy_{n+1}^0(t)/dt \\ &\;= e^t[A_3\widehat{C} + A_4\widehat{C} + A_3\widehat{x}_{n+1}^0(t) + A_4\widehat{x}_n^0(t) + B_3 y_\sigma(t) + B_4 y_\sigma(t) + B_3 y_{n+1}^0(t) + B_4\widehat{y}_n^0(t) + f_2(t)] \\ &\qquad\qquad = e^t[A_3\widehat{x}_{n+1}^0(t) + A_4\widehat{x}_n^0(t) + B_3 y_{n+1}^0(t) + B_4\widehat{y}_n^0]. \end{aligned}$$

This system is inhomogeneous, and the vector function $\varepsilon_n dy_\sigma(t)/dt$ in it is vanishing (it tends to zero as $n \to \infty$). Let us prove that the solution to this system $\{x_n^0(t), y_n^0(t)\}^\top \to 0$ as $n \to \infty$. Let $\widehat{q} = q + \mathbf{O}(\varepsilon)$, $q = e^{-\beta_0\sigma}$, and $q < \widehat{q} < 1$. Without loss of generality, we assume that

$$\widehat{q} > \mu : \mu = \delta\widehat{q}, \;\; 0 < \delta < 1. \tag{2.5}$$

Using the constant variation formula ([9], P. 157), we can write the solution to the perturbed inhomogeneous system on any step. We have

$$\begin{aligned} w_{n+k,\varepsilon}(t) = \widehat{T}_{n+k-1}\widehat{T}_{n+k-2}\ldots\widehat{T}_n(w_n(s)) + \varepsilon_1\widehat{T}_{n+k-1}\widehat{T}_{n+k-2}\ldots\widehat{T}_{n+1}dy_\sigma(t)/dt \\ + \varepsilon_2\widehat{T}_{n+k-1}\widehat{T}_{n+k-2}\ldots\widehat{T}_{n+2}dy_\sigma(t)/dt + \cdots + \varepsilon_k dy_\sigma(t)/dt. \end{aligned} \tag{2.6}$$

Here $\widehat{T}_n$ is the shift operator similar to (1.19) acting in the Banach space $C_{2m}[0,\tau]$ of continuous functions ([5], P. 124). In this space the solution to the unperturbed (linear) system is also representable in the operator form. Asymptotic properties of this operator [4] are analogous to properties of the operator T_{n,τ_n}. Namely, the product of these operators (as follows from Theorem 1) satisfies the bound

$$\Big\| \prod_{j=N}^{N+k} \widehat{T}_j w_{N-1}(s) \Big\| \le \widehat{L}_2\widehat{q}^k(\|x^{N-1}(t)\|_\tau + \|y_{N-1}(\tau)\|_\tau), \tag{2.7}$$

$$w_{N-1}^{(\top)}(t) = \{x^{N-1}(t), y^{N-1}(t)\}, \;\; \widehat{L}_2 \ge 1, \;\; k = 1, 2, \ldots$$

These operators are also uniformly bounded. Correlations (2.6) and (2.7) imply the inequality

$$\begin{aligned} \max_{0\le t\le\sigma}\big(\|x_{n+k}^0(t)\| + \|y_{n+k}^0(t)\|\big) \le \widehat{L}_2\widehat{q}^k \max_{0\le t\le\sigma}\big(\|x_n^0(t)\| + \|y_n^0(t)\|\big) + \widehat{L}_2\widehat{q}^{k-1}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| \\ + \widehat{L}_2\varepsilon_1\widehat{q}^{k-2}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| + \widehat{L}_2\varepsilon_2\widehat{q}^{k-3}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| + \cdots + \widehat{L}_2\varepsilon_{k-1}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\|. \end{aligned}$$

Hence, taking into account (2.5), we obtain

$$\begin{aligned} \max_{0\le t\le\sigma}\big(\|x_{n+k}^0(t)\| + \|y_{n+k}^0(t)\|\big) \le \widehat{L}_2\widehat{q}^k \max_{0\le t\le\sigma}\big(\|x_n^0(t)\| + \|y_n^0(t)\|\big) + \widehat{L}_2\widehat{q}^{k-1}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| \\ + \widehat{L}_2\widehat{q}^{k-1}\frac{\delta}{\theta_0}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| + \widehat{L}_2\widehat{q}^{k-1}\frac{\delta^2}{\theta_0}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| + \cdots + \widehat{L}_2\widehat{q}^{k-1}\frac{\delta^{k-1}}{\theta_0}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\| \\ \le \widehat{L}_2\widehat{q}^k \max_{0\le t\le\sigma}(\|x_n^0(t)\| + \|y_n^0(t)\|) + \frac{\widehat{L}_2\widehat{q}^{k-1}}{\theta_0(1-\delta)}\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\|. \end{aligned}$$

Since values $\max_{0\le t\le\sigma}\big(\|x_n^0(t)\| + y_n^0(t)\|\big)$, $\max_{0\le t\le\sigma}\|dy_\sigma(t)/dt\|$ are bounded, we conclude that

$$\max_{0\le t\le\sigma}\big(\|x_{n+k}^0(t)\| + \|y_{n+k}^0(t)\|\big) \to 0 \;\text{ as }\; k \to \infty.$$

$\square$

Remark. Condition 2) of Theorem 2 may be fulfilled accurate to a vanishing vector function.

3. AN EXAMPLE

Consider the asymptotic behavior of a fourth-order system with the following parameters: the delay $\sigma = 1$, matrices A_j and B_j, $j = 1, \dots, 4$, with components

$$A_1 = \begin{pmatrix} \sin pt & \cos pt \\ \sin pt & -\cos pt \end{pmatrix}, \quad A_2 = \begin{pmatrix} -\cos pt & \sin pt \\ \sin pt & \sin pt \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0.4 & 0.1 \\ 0 & 0.4 \end{pmatrix},$$
$$B_1 = \begin{pmatrix} 4\cos pt & -\cos pt \\ -0.5\cos pt & -\sin pt \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0.3\cos pt & -0.5\cos pt \\ \sin pt & \sin pt \end{pmatrix},$$
$$B_3 = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}, \tag{3.1}$$

and the zero matrix A_3. Here $p = 2\pi/\sigma$. Furthermore, the vector function $f_2(t) = \{\cos^3 pt, \sin^3 pt\}^\top$,

$$f_1(t) = \begin{pmatrix} 4\cos pt + 0.3\cos pt & -1.5\cos pt \\ -0.5\cos pt + \sin pt & 0 \end{pmatrix} * \begin{pmatrix} -0.602 & -0.422 \\ -0.606 & -0.452 \end{pmatrix}$$
$$* \left[\begin{pmatrix} 0.4 & 0.1 \\ 0 & 0.4 \end{pmatrix} * \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} \cos^3 pt \\ \sin^3 pt \end{pmatrix} \right] - \begin{pmatrix} 4\cos pt + 0.3\cos pt & -1.5\cos pt \\ -0.5\cos pt + \sin pt & 0 \end{pmatrix} * \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} -0.217\cos pt - 1.68\cos^4 pt - 1.137\cos pt\sin^3 pt \\ 0.849\cos pt - 1.699\sin pt + 0.301\cos^4 pt - 0.62\cos^3 pt\sin pt + 0.211\cos pt\sin^3 pt - 0.422\sin^4 pt \end{pmatrix}. \tag{3.2}$$

Evidently, the vector function $f_1(t)$ satisfies condition 2) of Theorem 2. Asymptotic properties of the corresponding homogeneous system (more precisely, the corresponding first approximation system) are described rather thoroughly in [3]; namely, it is shown that the first approximation system is exponentially stable and (as follows from Theorem 1) so is the corresponding homogeneous system. Finally, the fourth-order system described by equalities (3.1) and (3.2) has an asymptotically periodic solution in the form

$$x_\sigma = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad y_\sigma = \begin{pmatrix} -0.9 - 0.067\cos^3 pt - 0.417\sin^3 pt \\ -0.1 - 0.067\cos^3 pt - 0.017\sin^3 pt \end{pmatrix}.$$

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